

Distorting General Relativity: Gravity's Rainbow and $f(R)$ theories at work

Remo Garattini*

*Università degli Studi di Bergamo, Facoltà di Ingegneria,
Viale Marconi 5, 24044 Dalmine (Bergamo) Italy and
I.N.F.N. - sezione di Milano, Milan, Italy.*

We compute the Zero Point Energy in a spherically symmetric background combining the high energy distortion of Gravity's Rainbow with the modification induced by a $f(R)$ theory. Here $f(R)$ is a generic analytic function of the Ricci curvature scalar R in 4D and in 3D. The explicit calculation is performed for a Schwarzschild metric. Due to the spherically symmetric property of the Schwarzschild metric we can compare the effects of the modification induced by a $f(R)$ theory in 4D and in 3D. We find that the final effect of the combined theory is to have finite quantities that shift the Zero Point Energy. In this context we setup a Sturm-Liouville problem with the cosmological constant considered as the associated eigenvalue. The eigenvalue equation is a reformulation of the Wheeler-DeWitt equation which is analyzed by means of a variational approach based on gaussian trial functionals. With the help of a canonical decomposition, we find that the relevant contribution to one loop is given by the graviton quantum fluctuations around the given background. A final discussion on the connection of our result with the observed cosmological constant is also reported.

I. INTRODUCTION

In recent years many efforts have been done to modify General Relativity under different aspects. One of the main reasons is the search for a satisfying model of inflation and from another point of view try to find a *Quantum Gravity theory* which is still lacking. From one side, some modifications have their foundation in a class of theories termed *Extended Theories of Gravity* (ETG) which have become a sort of paradigm in the study of gravitational interaction based on corrections and enlargements of the Einstein scheme. The paradigm consists, essentially, in adding higher-order curvature invariants and non-minimally coupled scalar fields into dynamics resulting from the effective action of Quantum Gravity [1]. Such corrective terms seem to be unavoidable if we want to obtain the effective action of Quantum Gravity on scales closed to the Planck length [2]. Therefore terms of the form \mathcal{R}^2 , $\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}$, $\mathcal{R}^{\mu\nu\alpha\beta}\mathcal{R}_{\mu\nu\alpha\beta}$, $\mathcal{R}\Box\mathcal{R}$, or $\mathcal{R}\Box^k\mathcal{R}$ have to be added to the effective Lagrangian of gravitational field when quantum corrections are considered. To this purpose one should consider the possibility that the Hilbert - Einstein Lagrangian, linear in the scalar curvature \mathcal{R} , should be generalized into a generic function $f(\mathcal{R})$ ¹. Of course this modification will have some effects not only on large scale structure of space time, but even at small scales where quantum effects come into play. On the other side, one can distort space time introducing a modification which activates when the Planck scale energy is approached. Such a modification in its simplest form is known as *Doubly Special Relativity* (DSR)[5, 6]. One of the characterizing DSR effects is that the usual dispersion relation of a massive particle of mass m is modified into the following expression

$$E^2 g_1^2(E/E_P) - p^2 g_2^2(E/E_P) = m^2, \quad (1)$$

where $g_1(E/E_P)$ and $g_2(E/E_P)$ are two arbitrary functions which have the following property

$$\lim_{E/E_P \rightarrow 0} g_1(E/E_P) = 1 \quad \text{and} \quad \lim_{E/E_P \rightarrow 0} g_2(E/E_P) = 1. \quad (2)$$

The low energy limit (2) ensures that the usual dispersion relation is recovered. An immediate generalization to a curved background led Magueijo and Smolin[7] to introduce the idea of *Gravity's Rainbow* where a one parameter family of equations

$$G_{\mu\nu}(E) = 8\pi G(E) T_{\mu\nu}(E) + g_{\mu\nu} \Lambda(E), \quad (3)$$

replaces the ordinary Einstein's Field Equations. The meaning of $G(E)$ is represented by an energy dependent Newton's constant, defined so that $G(0)$ is the low-energy Newton's constant and similarly we have an energy dependent

*Electronic address: Remo.Garattini@unibg.it

¹ For a recent review, see Refs.[3, 4].

cosmological constant $\Lambda(E)$, defined so that $\Lambda(0)$ is the low-energy cosmological constant. Solutions of (3) have been found in Ref.[7]. One of these is the distorted Schwarzschild solution

$$ds^2 = - \left(1 - \frac{2MG(0)}{r} \right) \frac{dt^2}{g_1^2(E/E_P)} + \frac{dr^2}{\left(1 - \frac{2MG(0)}{r} \right) g_2^2(E/E_P)} + \frac{r^2}{g_2^2(E/E_P)} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4)$$

whose generalization leads to the following *rainbow* version of a spherically symmetric line element

$$ds^2 = - \frac{N^2(r)}{g_1^2(E/E_P)} dt^2 + \frac{dr^2}{\left(1 - \frac{b(r)}{r} \right) g_2^2(E/E_P)} + \frac{r^2}{g_2^2(E/E_P)} (d\theta^2 + \sin^2 \theta d\phi^2). \quad (5)$$

N is known as the lapse function and $b(r)$ is subject to the only condition $b(r_t) = r_t$. Motivated by the previous results obtained in $f(\mathcal{R})$ theories discussed in Ref.[8] and in Gravity's Rainbow obtained in² Refs.[9–11], in this paper we address the problem of computing Zero Point Energy (ZPE) combining the two modifications. It is known that ZPE calculations are affected by divergences that usually are kept under control with the help of a regularization and renormalization procedure. From this side, with an appropriate choice of the functions $g_1(E/E_P)$ and $g_2(E/E_P)$, Gravity's Rainbow seems to offer a method which avoids the usual regularization and renormalization procedure. Thus, while Gravity's Rainbow comes into play at Planckian scales, $f(\mathcal{R})$ theories are related to the large scale structure of space time, even if effects on short scales can be relevant, especially for ZPE evaluations[8]. In ordinary gravity the computation of ZPE for quantum fluctuations of the *pure gravitational field* can be extracted by rewriting the Wheeler-DeWitt equation (WDW)[13] in a form which looks like an expectation value computation[14]. We remind the reader that the WDW equation is the quantum version of the classical constraint which guarantees the invariance under time reparametrization. Its derivation is a consequence of the Arnowitt-Deser-Misner (\mathcal{ADM}) decomposition[17] of space time based on the following line element

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = (-N^2 + N_i N^i) dt^2 + 2N_j dt dx^j + g_{ij} dx^i dx^j, \quad (6)$$

where N is the *lapse* function and N_i the *shift* function. In terms of the \mathcal{ADM} variables, the four dimensional scalar curvature \mathcal{R} can be decomposed in the following way

$$\mathcal{R} = R + K_{ij} K^{ij} - (K)^2 - 2\nabla_\mu (K u^\mu + a^\mu), \quad (7)$$

where

$$K_{ij} = -\frac{1}{2N} [\partial_t g_{ij} - N_{i|j} - N_{j|i}] \quad (8)$$

is the second fundamental form, $K = g^{ij} K_{ij}$ is its trace, R is the three dimensional scalar curvature and \sqrt{g} is the three dimensional determinant of the metric. The last term in (7) represents the boundary terms contribution where the four-velocity u^μ is the timelike unit vector normal to the spacelike hypersurfaces ($t=\text{constant}$) denoted by Σ_t and $a^\mu = u^\alpha \nabla_\alpha u^\mu$ is the acceleration of the timelike normal u^μ . Thus

$$\mathcal{L}[N, N_i, g_{ij}] = \sqrt{-{}^4g} \mathcal{R} = \frac{N}{2\kappa} \sqrt{g} [K_{ij} K^{ij} - K^2 + R - 2\nabla_\mu (K u^\mu + a^\mu)] \quad (9)$$

represents the gravitational Lagrangian density where $\kappa = 8\pi G$ with G the Newton's constant. The WDW equation with the cosmological term included is described by

$$\mathcal{H}\Psi = \left[(2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (R - 2\Lambda) \right] \Psi = 0, \quad (10)$$

where G_{ijkl} is the super-metric and Λ is the cosmological constant. Note that $\mathcal{H} = 0$ represents one of the classical constraints. The other one is the invariance by spatial diffeomorphism. The form of \mathcal{H} is easily extracted with a Legendre transformation and with the help of the conjugate super-momentum defined as

$$\pi^{ij} = \frac{\delta \mathcal{L}}{\delta (\partial_t g_{ij})} = (g^{ij} K - K^{ij}) \frac{\sqrt{g}}{2\kappa}. \quad (11)$$

² Application of Gravity's Rainbow to discuss the traversability of wormholes can be found in Ref.[12].

Formally, the WDW equation can be transformed into an eigenvalue equation if we multiply Eq.(10) by $\Psi^*[g_{ij}]$ and functionally integrate over the three spatial metric g_{ij} . What we obtain is[14]³

$$\frac{1}{V} \frac{\int \mathcal{D}[g_{ij}] \Psi^*[g_{ij}] \int_{\Sigma} d^3x \hat{\Lambda}_{\Sigma} \Psi[g_{ij}]}{\int \mathcal{D}[g_{ij}] \Psi^*[g_{ij}] \Psi[g_{ij}]} = \frac{1}{V} \frac{\langle \Psi | \int_{\Sigma} d^3x \hat{\Lambda}_{\Sigma} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda}{\kappa}, \quad (12)$$

where we have also integrated over the hypersurface Σ and we have defined

$$V = \int_{\Sigma} d^3x \sqrt{g} \quad (13)$$

as the volume of the hypersurface Σ with

$$\hat{\Lambda}_{\Sigma} = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \sqrt{g} R / (2\kappa). \quad (14)$$

In this form, Eq.(12) can be used to compute ZPE provided that Λ/κ be considered as an eigenvalue of $\hat{\Lambda}_{\Sigma}$, namely the WDW equation is transformed into an expectation value computation. Nevertheless, solving Eq.(109) is a quite impossible task, therefore we are oriented to use a variational approach with trial wave functionals. The related boundary conditions are dictated by the choice of the trial wave functionals which, in our case, are of the Gaussian type: this choice is justified by the fact that ZPE should be described by a good candidate of the “*vacuum state*”. However different types of wave functionals correspond to different boundary conditions. It is better to observe that the obtained eigenvalue Λ/κ , it is far to be a constant, rather it will be dependent on some parameters like the mass M and the radial coordinate r for the Schwarzschild case and therefore it will be considered more like a “*dynamical cosmological constant*” evolving in r and M instead of a temporal parameter t . This is not a novelty, almost all the inflationary models try to substitute a cosmological constant Λ with some fields that changes with time. In this case, it is the gravity itself that gives a dynamical aspect to the “*cosmological constant* Λ ”, or more correctly the ZPE Λ/κ , without introducing any kind of external field but only quantum fluctuations. It remains to say that the additional distortion due to a $f(\mathcal{R})$ metric can be examined from two similar, but different point of view. Indeed for a spherically symmetric background, thanks to the \mathcal{ADM} decomposition one finds the identity $\mathcal{R} = R$ as can be seen from Eq.(7). Thus in this paper we will consider the effects on a ZPE computation of both a $f(\mathcal{R})$ theory as well as the ones of a $f(R)$ theory, namely a sub-class of the full covariant $f(\mathcal{R})$ models. It is relevant to say that models of $f(R)$ theories has been considered recently in the context of the Hořava-Lifshitz gravity[18], where the symmetry between space and time is explicitly broken. However both of the two formulations joined to Gravity’s Rainbow have to pass some tests. One of these is the Minkowski limit: in other words when the gravitational field is switched off, the induced cosmological constant must vanish. An example of this behavior can be found in Yang-Mills theory where the energy density for an $SU(2)$ massless gluon field in a constant color magnetic field can be written[19]

$$\frac{E}{V} = \frac{1}{2} H^2 + \frac{11}{48\pi^2} (eH)^2 \ln \left(\frac{eH}{\mu^2} - \frac{1}{2} \right). \quad (15)$$

This expression has a minimum away from the point $H = 0$, namely

$$eH_{\min} = \mu^2 \exp \left(-\frac{24\pi^2}{11e^2} \right) \quad (16)$$

and when $H \rightarrow 0$, $E/V \rightarrow 0$ that it means that in absence of an external field, the energy density is absent. The transposition to the gravitational field even if it is not immediate, it is pertinent: the gravitational field is the analogue of the Yang-Mills theory and the Schwarzschild metric is the analogous of the constant chromomagnetic field. It is interesting to note that even in our case, the minimum of the energy density or the maximum of the induced cosmological constant will be away from the point $M = 0$. It is important to say why one fixes the attention on the Schwarzschild metric: this is the only metric which is asymptotically flat depending only by one parameter, the mass M , which has zero classical energy density and therefore it is directly comparable with the Minkowski space. Also the Anti-de Sitter metric depends only on one parameter: Λ_{AdS} , but it is not asymptotically flat and this property makes the comparison between the two spaces a delicate procedure. The starting point of our analysis will be the line element (5), which will also be our cornerstone of the whole paper which is organized as follows. In section II,

³ An application of this calculation in the framework of Hořava-Lifshitz theory can be found in Ref.[15]

we report the results of Refs.[8, 9] and we derive separately the Hamiltonian constraint for a $f(\mathcal{R})$ theory without Gravity's Rainbow and the Hamiltonian constraint in presence of Gravity's Rainbow with the help of the background (5) but without a $f(\mathcal{R})$ theory term, in section III we compute the ZPE of quantum fluctuations with an $f(\mathcal{R})$ term around the background (5) and with the help of an appropriate choice of the functions $g_1(E/E_P)$ and $g_2(E/E_P)$, we will show that the UV divergences of ZPE disappear, in section IV we compute the ZPE of quantum fluctuations with an $f(R)$ term around the background (5) keeping under control the ZPE UV divergences. We summarize and conclude in section V. Units in which $\hbar = c = k = 1$ are used throughout the paper.

II. SETTING UP THE ZERO POINT ENERGY COMPUTATION WITH THE WHEELER-DEWITT EQUATION

The semi-classical procedure followed in this section relies heavily on the formalism outlined in Refs. [8, 16], where the graviton one loop contribution to a Schwarzschild background was computed, through a variational approach with Gaussian trial wave functionals [14]. Instead of using a zeta function regularization to deal with the divergences, we will use the arbitrariness of the rainbow's functions $g_1(E/E_P)$ and $g_2(E/E_P)$ avoiding therefore a renormalization procedure. Rather than reproduce the formalism, we shall refer the reader to Refs. [8, 9, 16] for details, when necessary. However, for self-completeness and self-consistency, we present here a brief outline of the formalism used.

A. The Wheeler-DeWitt Equation in the context of a $f(\mathcal{R})$ gravity theory

Let us consider now the Lagrangian density describing a generic $f(\mathcal{R})$ theory of gravity, namely

$$\mathcal{L} = \sqrt{-g} (f(\mathcal{R}) - 2\Lambda), \quad \text{with } f'' \neq 0, \quad (17)$$

where $f(\mathcal{R})$ is an arbitrary smooth function of the $4D$ scalar curvature and primes denote differentiation with respect to the scalar curvature. A cosmological term is added also in this case for the sake of generality. Obviously $f'' = 0$ corresponds to GR. To define the corresponding Hamiltonian one needs to define a further conjugate momentum⁴. To extract it from the Lagrangian density, we need to write \mathcal{L} in a form where the Lie derivative \mathcal{L}_n is explicit, namely

$$\mathcal{L}[N, N_i, g_{ij}] = \sqrt{-g} \mathcal{R} = \frac{N\sqrt{g}}{2\kappa} [R + K^2 - 3K_{ij}K^{ij} - 2N^{-1}h^{ij}N_{|ij} - 2h^{ij}\mathcal{L}_n K_{ij}]. \quad (18)$$

Note that in this form boundary terms do not appear. If we define

$$\mathcal{P}^{ij} = N^{-1} \frac{\partial \mathcal{L}}{\partial (\mathcal{L}_n K_{ij})} = -2\sqrt{g}g^{ij}f'(\mathcal{R}) \quad \implies \quad \mathcal{P} = -6\sqrt{g}f'(\mathcal{R}), \quad (19)$$

then following Ref.[8], one finds the generalized Hamiltonian density

$$\mathcal{H} = \frac{1}{2\kappa} \left[\frac{\mathcal{P}}{6} (R - 3K_{ij}K^{ij} + K^2) + \sqrt{g}U(\mathcal{P}) - \frac{1}{3}g^{ij}\mathcal{P}_{|ij} - 2\sqrt{g}K^{ij}K_{ij} \right], \quad (20)$$

where Λ is the cosmological constant and

$$U(\mathcal{P}) = \mathcal{R}f'(\mathcal{R}) - f(\mathcal{R}). \quad (21)$$

Assuming that $f'(\mathcal{R}) \neq 0$ and using the expression of the canonical momentum (11), the Hamiltonian constraint becomes

$$\int_{\Sigma} d^3x \left\{ \left[(2\kappa) \left[h(\mathcal{R}) G_{ijkl} \pi^{ij} \pi^{kl} + \frac{2(f'(\mathcal{R}) - 1)\pi^2}{f'(\mathcal{R})4} \right] - \frac{\sqrt{g}}{2\kappa} \left(R - 2\Lambda - \frac{U(\mathcal{P})}{f'(\mathcal{R})} \right) \right] \right\} = 0, \quad (22)$$

where we have integrated over the hypersurface Σ and we have used Gauss theorem on the three-divergence. When $f(\mathcal{R}) = \mathcal{R}$, $U(\mathcal{P}) = 0$ and

$$h(\mathcal{R}) = 1 + \frac{2[f'(\mathcal{R}) - 1]}{f'(\mathcal{R})} \rightarrow 1 \quad (23)$$

⁴ See also Ref.[20] for technical details.

as it should be. The WDW equation $\mathcal{H}\Psi = 0$ is the quantum version of the classical constraint. By repeating the steps which have led to Eq.(12), we obtain⁵

$$\frac{\langle \Psi | \int_{\Sigma} d^3x \left[\hat{\Lambda}_{\Sigma, f(\mathcal{R})} \right] | \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda^{f(\mathcal{R})}}{\kappa}, \quad (24)$$

where

$$\hat{\Lambda}_{\Sigma, f(\mathcal{R})} = (2\kappa) h(\mathcal{R}) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} R, \quad (25)$$

and where

$$\Lambda^{f(\mathcal{R})} = \Lambda + \frac{1}{2V} \int_{\Sigma} d^3x \sqrt{g} \frac{\mathcal{R} f'(\mathcal{R}) - f(\mathcal{R})}{f'(\mathcal{R})}. \quad (26)$$

Separating the degrees of freedom to one loop in perturbations, one finds that the graviton contribution is

$$\begin{aligned} \hat{\Lambda}_{\Sigma, f(\mathcal{R})}^{(2), \perp} &= \frac{1}{4V} \int_{\Sigma} d^3x \sqrt{g} G^{ijkl} \left[(2\kappa) h(\mathcal{R}) K^{-1\perp}(x, x)_{ijkl} + \frac{1}{(2\kappa)} (\Delta_L^m K^{\perp}(x, x))_{ijkl} \right] \\ &= \frac{1}{4} \sum_{\tau} \sum_{i=1}^2 \left[(2\kappa) h(\mathcal{R}) \lambda_i(\tau) + \frac{E_i^2(\tau)}{(2\kappa) \lambda_i(\tau)} \right] = -\frac{\Lambda^{f(\mathcal{R})}}{\kappa}, \end{aligned} \quad (27)$$

where we have used the following representation for the propagator $K^{\perp}(x, x)_{iakl}$

$$K^{\perp}(\vec{x}, \vec{y})_{iakl} := \sum_{\tau} \frac{h_{ia}^{(\tau)\perp}(\vec{x}) h_{kl}^{(\tau)\perp}(\vec{y})}{2\lambda(\tau)} \quad (28)$$

and where $h_{ia}^{(\tau)\perp}(\vec{x})$ are the eigenfunctions of Δ_L^m defined as

$$(\Delta_L^m h^{\perp})_{ij} = (\Delta h^{\perp})_{ij} - 4R_i^k h_{kj}^{\perp} + R h_{ij}^{\perp}, \quad (29)$$

which is the modified Lichnerowicz operator where Δ_L is the Lichnerowicz operator whose expression is

$$(\Delta_L h)_{ij} = \Delta h_{ij} - 2R_{ikjl} h^{kl} + R_{ik} h_j^k + R_{jk} h_i^k \quad \Delta = -\nabla^a \nabla_a. \quad (30)$$

τ denotes a complete set of indices and $\lambda(\tau)$ are a set of variational parameters to be determined by the minimization of Eq.(37). By minimizing with respect to the variational function $\lambda_i(\tau)$, we obtain the total one loop energy density for TT tensors

$$-\frac{\Lambda^{f(\mathcal{R})}}{\kappa} = \sqrt{h(\mathcal{R})} \frac{1}{4} \sum_{\tau} \left[\sqrt{E_1^2(\tau)} + \sqrt{E_2^2(\tau)} \right]. \quad (31)$$

The above expression makes sense only for $E_i^2(\tau) > 0$. Writing explicitly the previous expression we obtain

$$-\frac{\Lambda}{\kappa} - \frac{1}{2\kappa V} \int_{\Sigma} d^3x \sqrt{g} \frac{\mathcal{R} f'(\mathcal{R}) - f(\mathcal{R})}{f'(\mathcal{R})} = \sqrt{h(\mathcal{R})} \frac{1}{4} \sum_{\tau} \left[\sqrt{E_1^2(\tau)} + \sqrt{E_2^2(\tau)} \right]. \quad (32)$$

To evaluate the right hand side of the previous expression, we need a regularization prescription. In Ref.[8] a zeta function regularization and a renormalization of the induced cosmological constant have been used. Here we will consider the distortion of the space-time due to Gravity's Rainbow as a regulator to give meaning to Eq.(32). To do this we need to show how Gravity's Rainbow enters into the WDW equation.

⁵ See Ref.[8] for details.

B. The Wheeler-DeWitt Equation distorted by Gravity's Rainbow

We refer the reader to Ref. [9] for details, even if a brief outline will be presented. The distorted metric (5) induces the following transformations on the canonical momentum π^{ij} , the three scalar curvature R and the super-metric G_{ijkl}

$$\begin{cases} K_{ij} \rightarrow g_1(E) \tilde{K}_{ij}/g_2^2(E) \\ K \rightarrow g_1(E) \tilde{K} \\ \pi^{ij} \rightarrow g_1(E) \tilde{\pi}^{ij}/g_2(E) \\ \pi \rightarrow g_1(E) \tilde{\pi}/g_2^3(E) \\ R \rightarrow g_2^2(E) \tilde{R} \\ G_{ijkl} \rightarrow \tilde{G}_{ijkl}/g_2(E) \end{cases}, \quad (33)$$

where the symbol “ \sim ” indicates the quantity computed in absence of rainbow's functions $g_1(E)$ and $g_2(E)$. For simplicity, we have set $E_P = 1$ in $g_1(E/E_P)$ and $g_2(E/E_P)$ throughout the paragraph. With the help of the set (33) we can write the Hamiltonian constraint for the $f(\mathcal{R}) = \mathcal{R}$ theory, namely the ordinary GR

$$\mathcal{H} = (2\kappa) \frac{g_1^2(E)}{g_2^3(E)} \tilde{G}_{ijkl} \tilde{\pi}^{ij} \tilde{\pi}^{kl} - \frac{\sqrt{\tilde{g}}}{2\kappa g_2(E)} \left(\tilde{R} - \frac{2\Lambda_c}{g_2^2(E)} \right) = 0, \quad (34)$$

and the corresponding vacuum expectation value (12) becomes

$$\frac{g_2^3(E)}{\tilde{V}} \frac{\langle \Psi | \int_{\Sigma} d^3x \tilde{\Lambda}_{\Sigma} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda}{\kappa}, \quad (35)$$

with

$$\tilde{\Lambda}_{\Sigma} = (2\kappa) \frac{g_1^2(E)}{g_2^3(E)} \tilde{G}_{ijkl} \tilde{\pi}^{ij} \tilde{\pi}^{kl} - \frac{\sqrt{\tilde{g}} \tilde{R}}{(2\kappa) g_2(E)} \quad (36)$$

Extracting the TT tensor contribution from Eq.(35), we find

$$\hat{\Lambda}_{\Sigma}^{\perp} = \frac{g_2^3(E)}{4\tilde{V}} \int_{\Sigma} d^3x \sqrt{\tilde{g}} \tilde{G}^{ijkl} \left[(2\kappa) \frac{g_1^2(E)}{g_2^3(E)} \tilde{K}^{-1\perp}(x, x)_{ijkl} + \frac{1}{(2\kappa) g_2(E)} \left(\tilde{\Delta}_L^m \tilde{K}^{\perp}(x, x) \right)_{ijkl} \right], \quad (37)$$

with the prescription that the corresponding eigenvalue equation transforms into the following way

$$\left(\hat{\Delta}_L^m h^{\perp} \right)_{ij} = E^2 h_{ij}^{\perp} \quad \rightarrow \quad \left(\tilde{\Delta}_L^m \tilde{h}^{\perp} \right)_{ij} = \frac{E^2}{g_2^2(E)} \tilde{h}_{ij}^{\perp} \quad (38)$$

in order to reestablish the correct way of transformation of the perturbation. The propagator $K^{\perp}(x, x)_{iakl}$ will transform as

$$K^{\perp}(\vec{x}, \vec{y})_{iakl} \rightarrow \frac{1}{g_2^4(E)} \tilde{K}^{\perp}(\vec{x}, \vec{y})_{iakl}. \quad (39)$$

Thus the total one loop energy density for the graviton for the distorted GR becomes

$$\frac{\Lambda}{8\pi G} = -\frac{1}{2} \sum_{\tau} g_1(E) g_2(E) \left[\sqrt{E_1^2(\tau)} + \sqrt{E_2^2(\tau)} \right]. \quad (40)$$

In the next section we will combine the effects of Gravity's Rainbow and of an $f(\mathcal{R})$ theory to see the effects on the induced cosmological constant.

III. GRAVITY'S RAINBOW AND $f(\mathcal{R})$ GRAVITY AT WORK

To combine the effects of Gravity's Rainbow and of an $f(\mathcal{R})$ theory, we need to know how the scalar curvature \mathcal{R} transforms with respect to the rules (33). We find

$$\mathcal{R} \rightarrow \mathcal{R}_{g_1 g_2} = g_2^2(E) \tilde{R} + g_1^2(E) \left(\tilde{K}_{ij} \tilde{K}^{ij} - \left(\tilde{K} \right)^2 - 2\nabla_{\mu} \left(\tilde{K} \tilde{u}^{\mu} + \tilde{a}^{\mu} \right) \right). \quad (41)$$

With the transformation of the scalar curvature \mathcal{R} available, the new graviton operator becomes

$$\hat{\Lambda}_{\Sigma, f(\mathcal{R}_{g_1 g_2})}^\perp = \frac{g_2^3(E)}{4\tilde{V}} \int_{\Sigma} d^3x \sqrt{\tilde{g}} \tilde{G}^{ijkl} \left[(2\kappa) \frac{g_1^2(E)}{g_2^3(E)} h(\mathcal{R}_{g_1 g_2}) \tilde{K}^{-1\perp}(x, x)_{ijkl} + \frac{1}{(2\kappa) g_2(E)} \left(\tilde{\Delta}_L^m \tilde{K}^\perp(x, x) \right)_{ijkl} \right]. \quad (42)$$

Plugging the form of the propagator into Eq.(42), we find

$$-\frac{\Lambda^{f(\mathcal{R}_{g_1 g_2})}}{\kappa} = \frac{1}{2} \sum_{\tau} \sum_{i=1}^2 g_2^3(E) \left[(2\kappa) \frac{g_1^2(E)}{g_2^3(E)} \lambda_i(\tau) + \frac{E_i^2(\tau)}{(2\kappa) g_2(E) \lambda_i(\tau)} \right] \quad (43)$$

and the minimization with respect to the variational function $\lambda_i(\tau)$ leads to

$$\frac{\Lambda^{f(\mathcal{R}_{g_1 g_2})}}{\kappa} = -\frac{1}{2} \sum_{\tau} \sqrt{h(\mathcal{R}_{g_1 g_2}) g_1(E) g_2(E)} \left[\sqrt{E_1^2(\tau)} + \sqrt{E_2^2(\tau)} \right], \quad (44)$$

where $\Lambda^{f(\mathcal{R}_{g_1 g_2})}$ is expressed by the Eq.(26). The above expression makes sense only for $E_i^2(\tau) > 0$. With the help of Regge and Wheeler representation[21], the eigenvalue equation (38) can be reduced to

$$\left[-\frac{d^2}{dx^2} + \frac{l(l+1)}{r^2} + m_i^2(r) \right] f_i(x) = \frac{E_{i,l}^2}{g_2^2(E)} f_i(x) \quad i = 1, 2, \quad (45)$$

where we have used reduced fields of the form $f_i(x) = F_i(x)/r$ and where we have defined two r-dependent effective masses $m_1^2(r)$ and $m_2^2(r)$

$$\begin{cases} m_1^2(r) = \frac{6}{r^2} \left(1 - \frac{b(r)}{r} \right) + \frac{3}{2r^2} b'(r) - \frac{3}{2r^3} b(r) \\ m_2^2(r) = \frac{6}{r^2} \left(1 - \frac{b(r)}{r} \right) + \frac{1}{2r^2} b'(r) + \frac{3}{2r^3} b(r) \end{cases} \quad (r \equiv r(x)). \quad (46)$$

In order to use the WKB approximation, from Eq.(45) we can extract two r-dependent radial wave numbers

$$k_i^2(r, l, \omega_{i,nl}) = \frac{E_{i,nl}^2}{g_2^2(E)} - \frac{l(l+1)}{r^2} - m_i^2(r) \quad i = 1, 2. \quad (47)$$

Using the 't Hooft method[22], we count the number of modes with frequency less than E_i , $i = 1, 2$. This is given approximately by

$$\tilde{g}(E_i) = \int_0^{l_{\max}} \nu_i(l, E_i) (2l+1) dl, \quad (48)$$

where $\nu_i(l, E_i)$, $i = 1, 2$ is the number of nodes in the mode with (l, E_i) , such that $(r \equiv r(x))$

$$\nu_i(l, E_i) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dx \sqrt{k_i^2(r, l, E_i)}. \quad (49)$$

Here it is understood that the integration with respect to x and l_{\max} is taken over those values which satisfy $k_i^2(r, l, E_i) \geq 0$, $i = 1, 2$. With the help of Eqs.(48, 49), Eq.(44) becomes

$$\frac{\Lambda^{f(\mathcal{R}_{g_1 g_2})}}{8\pi G} = -\frac{1}{\pi} \sum_{i=1}^2 \int_0^{+\infty} \sqrt{h(\mathcal{R}_{g_1 g_2}) E_i} g_1(E) g_2(E) \frac{d\tilde{g}(E_i)}{dE_i} dE_i, \quad (50)$$

where

$$\begin{aligned} \frac{d\tilde{g}(E_i)}{dE_i} &= \int \frac{\partial \nu(l, E_i)}{\partial E_i} (2l+1) dl = \frac{1}{\pi} \int_{-\infty}^{+\infty} dx \int_0^{l_{\max}} \frac{(2l+1)}{\sqrt{k_i^2(r, l, E)}} \frac{d}{dE_i} \left(\frac{E_i^2}{g_2^2(E)} - m_i^2(r) \right) dl \\ &= \frac{2}{\pi} \int_{-\infty}^{+\infty} dx r^2 \frac{d}{dE_i} \left(\frac{E_i^2}{g_2^2(E)} - m_i^2(r) \right) \sqrt{\frac{E_i^2}{g_2^2(E)} - m_i^2(r)} = \frac{4}{3\pi} \int_{-\infty}^{+\infty} dx r^2 \frac{d}{dE_i} \left(\frac{E_i^2}{g_2^2(E)} - m_i^2(r) \right)^{\frac{3}{2}}. \end{aligned} \quad (51)$$

Plugging expression (51) into Eq.(50) and dividing for a volume factor, we obtain

$$\begin{aligned} \frac{\Lambda^f(\mathcal{R}_{g_1 \ g_2})}{8\pi G} &= -\frac{1}{3\pi^2} \sum_{i=1}^2 \int_{E^*}^{+\infty} \sqrt{h(\mathcal{R}_{g_1 \ g_2})} E_i g_1(E) g_2(E) \frac{d}{dE_i} \sqrt{\left(\frac{E_i^2}{g_2^2(E)} - m_i^2(r)\right)^3} dE_i \\ &= -\frac{1}{\pi^2} \sum_{i=1}^2 \int_{E^*}^{+\infty} \sqrt{h(\mathcal{R}_{g_1 \ g_2})} E_i^2 g_1(E) \sqrt{\frac{E_i^2}{g_2^2(E)} - m_i^2(r)} d\left(\frac{E_i}{g_2(E)}\right) \end{aligned} \quad (52)$$

where E^* is the value which annihilates the argument of the root. In the previous equation, we have included an additional 4π factor coming from the angular integration and we have assumed that the effective mass does not depend on the energy E . Using Eq.(26), we find

$$\frac{\Lambda}{\kappa} = -\frac{1}{2\kappa V} \int_{\Sigma} d^3x \sqrt{g} \frac{\mathcal{R} f'(\mathcal{R}_{g_1 \ g_2}) - f(\mathcal{R}_{g_1 \ g_2})}{f'(\mathcal{R}_{g_1 \ g_2})} \quad (53)$$

$$-\frac{1}{\pi^2} \sum_{i=1}^2 \int_{E^*}^{+\infty} \sqrt{h(\mathcal{R}_{g_1 \ g_2})} E_i^2 g_1(E) \sqrt{\frac{E_i^2}{g_2^2(E)} - m_i^2(r)} d\left(\frac{E_i}{g_2(E)}\right). \quad (54)$$

A general solution for the previous expression is a difficult task. However, we can obtain enough information even if we fix the attention on some spherically symmetric backgrounds which have the following property

$$m_0^2(r) = m_2^2(r) = -m_1^2(r), \quad \forall r \in (r_t, r_1). \quad (55)$$

For example, the Schwarzschild background represented by the choice $b(r) = r_t = 2MG$ satisfies the property (55) in the range $r \in [r_t, 5r_t/2]$. Similar backgrounds are the Schwarzschild-de Sitter and Schwarzschild-Anti de Sitter. On the other hand other backgrounds, like dS, AdS and Minkowski have the property

$$m_0^2(r) = m_2^2(r) = m_1^2(r), \quad \forall r \in (r_t, \infty). \quad (56)$$

In this paper, we will fix our attention only on metrics which satisfy the condition (55) and in particular only for the Schwarzschild background. The spherical symmetry of the metric (5) allows to further reduce the scalar curvature in $4D$,

$$\mathcal{R}_{g_1 \ g_2} = g_2^2(E) \tilde{R} = 2g_2^2(E) \frac{b'(r)}{r^2} \quad (57)$$

where we have used the mixed Ricci tensor \tilde{R}_j^a whose components are:

$$\tilde{R}_j^a = \left\{ \frac{b'(r)}{r^2} - \frac{b(r)}{r^3}, \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3}, \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3} \right\}. \quad (58)$$

Thus Eq. (54) becomes

$$\begin{aligned} \frac{\Lambda}{\kappa} &= -\frac{1}{2\kappa V} \int_{\Sigma} d^3x \sqrt{g} \left[2g_2^2(E) \frac{b'(r)}{r^2} - \frac{f\left(2g_2^2(E) \frac{b'(r)}{r^2}\right)}{f'\left(2g_2^2(E) \frac{b'(r)}{r^2}\right)} \right] \\ &\quad - \frac{1}{\pi^2} \sum_{i=1}^2 \int_{E^*}^{+\infty} \sqrt{h\left(2g_2^2(E) \frac{b'(r)}{r^2}\right)} E_i^2 g_1(E) \sqrt{\frac{E_i^2}{g_2^2(E)} - m_i^2(r)} d\left(\frac{E_i}{g_2(E)}\right) \end{aligned} \quad (59)$$

and for the Schwarzschild background one finds a further simplification

$$\frac{\Lambda}{\kappa} = \frac{1}{2\kappa} \frac{f(0)}{f'(0)} - \frac{1}{\pi^2} (I_+ + I_-), \quad (60)$$

where

$$I_+ = \sqrt{3 - \frac{1}{f'(0)}} \int_0^{+\infty} E^2 g_1(E) \sqrt{\frac{E^2}{g_2^2(E)} + m_0^2(r)} d\left(\frac{E}{g_2(E)}\right) \quad (61)$$

and

$$I_- = \sqrt{3 - \frac{1}{f'(0)}} \int_{E^*}^{+\infty} E^2 g_1(E) \sqrt{\frac{E^2}{g_2^2(E)} - m_0^2(r)} d\left(\frac{E}{g_2(E)}\right), \quad (62)$$

where we have used the assumption that $f'(0) \neq 0$ and $b(r) = r_t = 2MG$. It is immediate to recognize that if

$$f'(0) = \frac{1}{3} \quad \implies \quad I_+ = I_- = 0 \quad (63)$$

and the quantum contribution disappears leaving

$$\Lambda = \frac{3}{2} f(0). \quad (64)$$

Of course, the disappearance of the quantum contribution to one loop means that we have to compute higher order contributions to the induced cosmological constant. On the other hand, when $f'(0) \neq 1/3$, we have to evaluate I_+ and I_- . To do calculations in practice, we need to specify the form of $g_1(E)$ and $g_2(E)$ in such a way I_+ and I_- be finite and condition (2) be satisfied. Since the general case is highly non trivial as one can deduce from expression (59) and it strongly depends on the form of $f(\mathcal{R}_{g_1 g_2})$, we fix our attention on some examples which can be examined in the context of the Schwarzschild metric. The examples we are going to discuss are:

a)

$$g_1\left(\frac{E}{E_P}\right) = (1 + c_2 \frac{E}{E_P}) \exp(-c_1 \frac{E^2}{E_P^2}) \quad g_2(E/E_P) = 1, \quad (65)$$

b)

$$g_1\left(\frac{E}{E_P}\right) = (1 + c_2 \frac{E}{E_P}) \exp(-c_1 \frac{E^2}{E_P^2}) \quad g_2(E/E_P) = 1 + c_3 \frac{E}{E_P} \quad (66)$$

and⁶

c)

$$g_2^{-2}(E/E_P) = g_1(E/E_P), \quad (69)$$

where for convenience we have reintroduced the dependence on E_P . All these cases have to pass the Minkowski limit test, namely in absence of a curved background one must reproduce a vanishing cosmological constant. This test furthermore allows to fix the value of $f(0)$ at least for the Schwarzschild case. Beginning with the case a), which has been studied in Ref.[9] for $f(\mathcal{R}) = \mathcal{R}$, we find that the computation of the integrals (61) and (62) leads to a finite

⁶ Equivalent proposals to models b) and c) from the ultraviolet point of view are, for example:

$$g_2\left(\frac{E}{E_P}\right) = \frac{1}{1 + \alpha \frac{E}{E_P} \tanh\left(\frac{E}{E_P}\right)} \quad (67)$$

and

$$g_2\left(\frac{E}{E_P}\right) = \frac{1}{1 + \alpha \frac{E}{E_P} \arctan\left(\frac{E}{E_P}\right)}. \quad (68)$$

value of the induced cosmological constant. Here, we report the asymptotic expansion for small and large x of $I_+ + I_-$ with the factor $\sqrt{3 - \frac{1}{f'(0)}}$ dropped, where

$$x = \sqrt{\frac{m_0^2(r)}{E_P^2}} = \sqrt{\frac{3MG}{r^3 E_P^2}}. \quad (70)$$

For large x , one gets

$$-\frac{1}{\pi^2 E_P^4} (I_+ + I_-) \simeq -\frac{(2c_2 c_1^{3/2} + \sqrt{\pi} c_1^2) x}{4\pi^2 c_1^{7/2}} - \frac{8c_2 c_1^{5/2} + 3\sqrt{\pi} c_1^3}{16\pi^2 c_1^{11/2} x} + \frac{3}{128\pi^2} \frac{16c_2 c_1^{7/2} + 5\sqrt{\pi} c_1^4}{c_1^{15/2} x^3} + O(x^{-4}), \quad (71)$$

while for small x we obtain

$$-\frac{1}{\pi^2 E_P^4} (I_+ + I_-) \simeq -\frac{4c_1^{5/2} + 3\sqrt{\pi} c_2 c_1^2}{4\pi^2 c_1^{9/2}} + O(x^3). \quad (72)$$

It is straightforward to see that if we set

$$c_2 = -\frac{\sqrt{c_1 \pi}}{2}, \quad (73)$$

the linear divergent term of the asymptotic expansion of expansion (71) disappears. Plugging the relationship (73) into (72), one finds

$$\frac{I_+ + I_-}{\pi^2 E_P^4} = \frac{3\pi - 8}{8\pi^2 c_1^2}. \quad (74)$$

Thus the induced cosmological constant (60) becomes when $x \rightarrow 0$

$$\frac{\Lambda}{\kappa} = \frac{1}{2\kappa} \frac{f(0)}{f'(0)} - E_P^4 \frac{3\pi - 8}{8\pi^2 c_1^2} \quad (75)$$

and using the freedom to fix the value of $f(0)$, we find⁷

$$f(0) = f'(0) 2E_P^2 \frac{3\pi - 8}{\pi c_1^2} \quad x = 0. \quad (76)$$

Therefore the appropriate $f(\mathcal{R})$ model should be defined with the following property

$$f(0) = \begin{cases} f'(0) 2E_P^2 (3\pi - 8) / (\pi c_1^2) & x = 0 \\ 0 & x > 0 \end{cases}. \quad (77)$$

Of course this definition works only for the Schwarzschild background. Note that when we are on the throat $r = 2MG$ ($G = E_P^{-2}$)

$$x = x_M = \frac{E_P}{2M} \sqrt{\frac{3}{2}}, \quad (78)$$

and $x \rightarrow \infty$ when $M \rightarrow 0$. However the parametrization (65) leads to a convergent result when $x \rightarrow \infty$ as indicated by the expression (71). Therefore we conclude that with the choice (77) we can reproduce the Minkowski limit for every value of M . As regards the case b), we can consider

$$\begin{cases} I_+ = \sqrt{3 - \frac{1}{f'(0)}} (I_{+,1} + I_{+,2}) \\ I_- = \sqrt{3 - \frac{1}{f'(0)}} (I_{-,1} + I_{-,2}) \end{cases}, \quad (79)$$

⁷ Note that in Ref.[9], the value of c_1 has been fixed to $1/4$ as suggested by Noncommutative theory[23].

where

$$\begin{cases} I_{+,1} = \int_0^{E_P} E^2 \sqrt{E^2 + m_0^2(r)} dE \\ I_{+,2} = \int_{E_P}^{+\infty} E^2 g_1(E) \sqrt{\frac{E^2}{g_2^2(E)} + m_0^2(r)} d\left(\frac{E}{g_2(E)}\right) \\ I_{-,1} = \int_{\sqrt{m_0^2(r)}}^{E_P} E^2 \sqrt{E^2 - m_0^2(r)} dE \\ I_{-,2} = \int_{E_P}^{+\infty} E^2 g_1(E) \sqrt{\frac{E^2}{g_2^2(E)} - m_0^2(r)} d\left(\frac{E}{g_2(E)}\right) \end{cases} . \quad (80)$$

$I_{+,1}$ and $I_{-,1}$ can be computed exactly. Indeed the result of the integration leads to

$$\begin{cases} I_{+,1} = \frac{1}{8} E_P^4 \left[2(1+x^2)^{\frac{3}{2}} - x^2 \sqrt{1+x^2} - x^4 \left(\ln(1+\sqrt{x^2+1}) - \ln \sqrt{x^2} \right) \right] \\ I_{-,1} = \frac{1}{8} E_P^4 \left[2(1-x^2)^{\frac{3}{2}} - x^2 \sqrt{1-x^2} - x^4 \left(\ln(1+\sqrt{1-x^2}) - \ln \sqrt{x^2} \right) \right] \end{cases} . \quad (81)$$

Concerning $I_{+,2}$ and $I_{-,2}$ we get

$$\begin{cases} I_{+,2} = \int_{E_P}^{+\infty} E^2 \left((1 + c_2 \frac{E}{E_P}) \exp(-c_1 \frac{E^2}{E_P^2}) \right) \sqrt{\left(\frac{E}{1+c_3 \frac{E}{E_P}} \right)^2 + m_0^2(r)} \frac{dE}{\left(1+c_3 \frac{E}{E_P} \right)^2} \\ I_{-,2} = \int_{E_P}^{+\infty} E^2 \left((1 + c_2 \frac{E}{E_P}) \exp(-c_1 \frac{E^2}{E_P^2}) \right) \sqrt{\left(\frac{E}{1+c_3 \frac{E}{E_P}} \right)^2 - m_0^2(r)} \frac{dE}{\left(1+c_3 \frac{E}{E_P} \right)^2} \end{cases} . \quad (82)$$

It is immediate to see that for a class of rainbow functions $g_2(E/E_P)$ increasing faster than E^2 , the integrand in $I_{-,2}$ can become imaginary and therefore leads to an imaginary induced cosmological constant. This is the main reason to fix the ideas on the choice made in (66). Note that in the range where $E \in [E_P, +\infty)$, one can write $g_2(E/E_P) \sim E_P/(c_3 E)$. Then using the definition (70), $I_{+,2}$ and $I_{-,2}$ can be easily calculated to give

$$\begin{cases} I_{+,2} = \frac{E_P^4}{2\sqrt{c_1} c_3^3} \sqrt{1 + c_3^2 x^2} \left[\sqrt{\pi} (1 - \operatorname{erf}(\sqrt{c_1})) + c_2 \frac{e^{-c_1}}{\sqrt{c_1}} \right] \\ I_{-,2} = \frac{E_P^4}{2\sqrt{c_1} c_3^3} \sqrt{1 - c_3^2 x^2} \left[\sqrt{\pi} (1 - \operatorname{erf}(\sqrt{c_1})) + c_2 \frac{e^{-c_1}}{\sqrt{c_1}} \right] \end{cases} , \quad (83)$$

where $\operatorname{erf}(x)$ is the error function and where we have used the following relationship

$$\int_{E_P}^{+\infty} \left[(1 + c_2 \frac{E}{E_P}) \exp(-c_1 \frac{E^2}{E_P^2}) \right] dE = \frac{E_P \sqrt{\pi}}{2\sqrt{c_1}} (1 - \operatorname{erf}(\sqrt{c_1})) + c_2 \frac{e^{-c_1}}{2c_1} . \quad (84)$$

Summing up the various contributions, Eq.(60) becomes

$$\begin{aligned} \frac{\Lambda}{\kappa} = & \frac{1}{2\kappa} \frac{f(0)}{f'(0)} - \frac{E_P^4}{8\pi^2} \sqrt{3 - \frac{1}{f'(0)}} \left\{ 2(1+x^2)^{\frac{3}{2}} - x^2 \left(\sqrt{1+x^2} + \sqrt{1-x^2} \right) - x^4 \ln \left(\frac{1}{x} + \sqrt{1 + \frac{1}{x^2}} \right) \right. \\ & \left. + 2(1-x^2)^{\frac{3}{2}} - x^4 \ln \left(\frac{1}{x} + \sqrt{1 - \frac{1}{x^2}} \right) + \frac{8}{c_3^3} \left(\sqrt{\pi} (1 - \operatorname{erf}(1/2)) + \frac{2c_2}{\sqrt{e}} \right) \left[\sqrt{1 + c_3^2 x^2} + \sqrt{1 - c_3^2 x^2} \right] \right\} , \end{aligned} \quad (85)$$

where we have fixed $c_1 = 1/4$, like in Ref.[9]. Now we have to verify the Minkowski limit with the computation of

$$\lim_{x \rightarrow 0} \frac{\Lambda}{\kappa} = \frac{1}{2\kappa} \frac{f(0)}{f'(0)} - \frac{E_P^4}{8\pi^2} \sqrt{3 - \frac{1}{f'(0)}} \left[4 + \frac{16}{c_3^3} \left(\sqrt{\pi} (1 - \operatorname{erf}(1/2)) + \frac{2c_2}{\sqrt{e}} \right) \right] = 0, \quad (86)$$

where x has been defined in (70). If $f(0) = 0$, we have to impose that

$$c_2 = -2\sqrt{e} \left(1 + \frac{4}{c_3^3} \sqrt{\pi} (1 - \operatorname{erf}(1/2)) \right) \quad (87)$$

to obtain the vanishing of the induced cosmological constant (85). Otherwise if

$$f(0) \neq 0 \quad \Rightarrow \quad f(0) = \frac{2E_P^2}{\pi} \sqrt{3f'^2(0) - f'(0)} \left[4 + \frac{16}{c_3^3} \left(\sqrt{\pi} (1 - \operatorname{erf}(1/2)) + \frac{2c_2}{\sqrt{e}} \right) \right]. \quad (88)$$

All we have found for the case b) is valid for $r > 2MG$. However when we reach the throat $r_t = 2MG$, Eq.(70) has to be substituted with Eq.(78) where for $M \rightarrow 0$, $x \rightarrow \infty$. Therefore we conclude that the case b) cannot be considered as a good description of the induced cosmological constant for every value of M . As regards the case c), I_+ and I_- reduce to

$$I_+ = \sqrt{3 - \frac{1}{f'(0)}} \int_0^{+\infty} \left(\frac{E}{g_2(E/E_P)} \right)^2 \sqrt{\frac{E^2}{g_2^2(E/E_P)} + m_0^2(r)} d \left(\frac{E}{g_2(E/E_P)} \right) \quad (89)$$

and

$$I_- = \sqrt{3 - \frac{1}{f'(0)}} \int_{E^*}^{+\infty} \left(\frac{E}{g_2(E/E_P)} \right)^2 \sqrt{\frac{E^2}{g_2^2(E/E_P)} - m_0^2(r)} d \left(\frac{E}{g_2(E/E_P)} \right). \quad (90)$$

With the help of the auxiliary variable

$$z(E/E_P) = \frac{E/E_P}{g_2(E/E_P)}, \quad (91)$$

we find that Eq.(60) becomes:

$$\frac{\Lambda}{8\pi G} = \frac{1}{2\kappa} \frac{f(0)}{f'(0)} - \frac{I(z_\infty, x)}{8\pi^2} E_P^4 \sqrt{3 - \frac{1}{f'(0)}}, \quad (92)$$

where $z_\infty = \lim_{E \rightarrow \infty} z(E/E_P)$ and where we have used Eq.(70) to obtain

$$\begin{aligned} I(z_\infty, x) = & \left[z_\infty (2z_\infty^2 - x^2) \sqrt{z_\infty^2 - x^2} - x^4 \ln \left(\frac{z_\infty}{x} + \sqrt{\frac{z_\infty^2}{x^2} - 1} \right) \right. \\ & \left. + z_\infty (2z_\infty^2 + x^2) \sqrt{z_\infty^2 + x^2} - x^4 \ln \left(\frac{z_\infty}{x} + \sqrt{\frac{z_\infty^2}{x^2} + 1} \right) \right]. \end{aligned} \quad (93)$$

For the reasons discussed in case b), we adopt the proposal (66) and the limit

$$z_\infty = \lim_{E \rightarrow \infty} z\left(\frac{E}{E_P}\right) = \lim_{E \rightarrow \infty} \frac{E/E_P}{1 + c_3 E/E_P} = \frac{1}{c_3} \quad (94)$$

becomes a constant. From Eq.(92) to verify if the Minkowski limit is satisfied for the case $r > 2MG$, we find

$$\lim_{M \rightarrow 0} \frac{\Lambda}{8\pi G} = \frac{1}{2\kappa} \frac{f(0)}{f'(0)} - \frac{E_P^4}{8\pi^2 c_3^4} \sqrt{3 - \frac{1}{f'(0)}} I(1, y). \quad (95)$$

where $y = c_3 x$ and $I(1, y = 0) = 4$. Thus by defining

$$f(0) = \frac{f'(0) 8E_P^2}{\pi c_3^4} \sqrt{3 - \frac{1}{f'(0)}}, \quad (96)$$

Eq.(95) becomes

$$\frac{\Lambda}{8\pi G} = \frac{E_P^4}{8\pi^2 c_3^4} \sqrt{3 - \frac{1}{f'(0)}} [4 - I(1, y)]. \quad (97)$$

In this way the Minkowski limit is satisfied. A particular attention has to be considered for the case $r = 2MG$. As shown in Eq.(78), $y \rightarrow \infty$ when $M \rightarrow 0$. From the limit (94) and the function (93), one gets

$$I(z_\infty, x) = I\left(\frac{1}{c_3}, \frac{E_P}{2M}\sqrt{\frac{3}{2}}\right) \quad (98)$$

which becomes imaginary when $M < c_3 E_P \sqrt{3}/\sqrt{8}$. To avoid this drawback, we can use the arbitrariness of c_3 by imposing that $I(a, b)$ be equal to

$$I\left(\frac{E_P}{2M}\sqrt{\frac{3}{2}}, \frac{E_P}{2M}\sqrt{\frac{3}{2}}\right) = \frac{9E_P^4}{64M^4} \left[3\sqrt{2} - \ln(1 + \sqrt{2})\right] \quad \text{when} \quad M \leq \frac{E_P}{2}\sqrt{\frac{3}{2}}c_3. \quad (99)$$

In this way, if we impose the Minkowski limit, Eq.(95) allows to define

$$f(0) = f'(0) \frac{9E_P^2}{32\pi} \left(\frac{E_P}{M}\right)^4 \sqrt{3 - \frac{1}{f'(0)}} \left(3\sqrt{2} - \ln(1 + \sqrt{2})\right). \quad (100)$$

Note that in this case we cannot fix $f(0) = 0$, because there is not an appropriate choice of the parameters such that the induced cosmological constant vanishes. Plugging the value of $f(0)$ of (100) into Eq.(95) leads to

$$\frac{\Lambda}{8\pi G} = \frac{E_P^4}{8\pi^2} \sqrt{3 - \frac{1}{f'(0)}} \left[\frac{9}{64} \left(\frac{E_P}{M}\right)^4 \left(3\sqrt{2} - \ln(1 + \sqrt{2})\right) - I\left(\frac{1}{c_3}, \frac{E_P}{2M}\sqrt{\frac{3}{2}}\right) \right] \quad (101)$$

and the Minkowski limit is reached when $M \rightarrow 0$.

IV. GRAVITY'S RAINBOW AND $f(R)$ GRAVITY AT WORK

In the previous section we have considered the effects of Gravity's Rainbow combined with those of an $f(\mathcal{R})$ theory on the computation of the induced cosmological constant. Since we have fixed our ideas on a Schwarzschild background, the original $f(\mathcal{R})$ theory has considerably reduced to be a function of three scalar curvature R . For this reason, in this section we will consider a theory that modifies only the three dimensional spatial space. We are therefore led to consider an arbitrary smooth function of the three dimensional scalar curvature $f(R)$ combined with an ordinary General Relativity four dimensional scalar curvature \mathcal{R} , namely

$$f(\mathcal{R}) = \mathcal{R} + f(R). \quad (102)$$

In the \mathcal{ADM} formulation, the Lagrangian density becomes

$$\mathcal{L} = \frac{N}{2\kappa} \sqrt{g} f(\mathcal{R}) = \frac{N}{2\kappa} \sqrt{g} \left[R + K_{ij} K^{ij} - (K)^2 - 2\nabla_\mu (K u^\mu + a^\mu) + f(R) \right], \quad (103)$$

where we have used the decomposition (7). Differently from the $f(\mathcal{R})$ model where

$$\mathcal{P}^{ij} = -2\sqrt{g} g^{ij} f'(\mathcal{R}), \quad (104)$$

in a $f(R)$ model, \mathcal{P}^{ij} is absent and the Hamiltonian can be computed in an ordinary way. Therefore the rôle of $f(R)$ is to shift the value of R . One simply obtains

$$\mathcal{H} = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (f(R) + R - 2\Lambda), \quad (105)$$

where a cosmological term has been introduced for a sake of generality. For a background of the form (5) in the low energy limit, the Hamiltonian constraint reduces to

$$\mathcal{H} = f\left(2\frac{b'(r)}{r^2}\right) + 2\frac{b'(r)}{r^2} - 2\Lambda = 0. \quad (106)$$

Solutions of the classical constraint depend on a case to case. However for the simple case of $f(R) = \text{const.}$, a solution is represented by a Schwarzschild-de Sitter or Schwarzschild-Anti de Sitter metric depending on the sign of $f(R) - 2\Lambda$. In particular, for the Schwarzschild solution, we find that the classical constraint becomes

$$f(0) = 2\Lambda. \quad (107)$$

The quantization procedure is obtained with the help of the modified WDW equation

$$\mathcal{H}\Psi = \left[(2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (f(R) + R - 2\Lambda) \right] \Psi = 0. \quad (108)$$

Following the procedure which has led to Eq.(12), we can write [8, 16]

$$\frac{1}{V} \frac{\langle \Psi | \int_{\Sigma} d^3x \hat{\Lambda}_{\Sigma} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda^{f(R)}}{\kappa}, \quad (109)$$

where we have used Eq.(13) and Eq.(14). In this form, Eq.(109) can be used to compute ZPE provided that $\Lambda^{f(R)}/\kappa$ be considered as an eigenvalue of $\hat{\Lambda}_{\Sigma}$, where in this case

$$\Lambda^{f(R)} = \Lambda - \frac{1}{2V} \frac{\langle \Psi | \int_{\Sigma} d^3x \sqrt{g} f(R) | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \Lambda - \frac{1}{2V} \int_{\Sigma} d^3x \sqrt{g} f(R). \quad (110)$$

Note that, differently from Eq.(26), the distorted Λ does not depend on derivatives of $f(R)$. However the evaluation of the l.h.s. of (109) leads to the usual one loop divergences. These can be kept under control, with the help of the Rainbow's functions considered in the previous section. We know that the line element (5) induces a modification of the scalar curvature \mathcal{R} leading to $\mathcal{R}_{g_1 g_2}$ described by Eq.(41). In this way, the Lagrangian density (103) changes into

$$\mathcal{L} \rightarrow \frac{\tilde{N}\sqrt{\tilde{g}}}{g_1(E)g_2^3(E)} \left\{ g_2^2(E) \left[\tilde{R} + \tilde{K}_{ij} \tilde{K}^{ij} - (\tilde{K})^2 \right] - 2\nabla_{\mu} \left(g_1^2(E) (\tilde{K} \tilde{u}^{\mu} + \tilde{a}^{\mu}) \right) + f(g_2^2(E) \tilde{R}) \right\} \quad (111)$$

and it is straightforward to see that the modified classical constraint becomes

$$\mathcal{H}_m = (2\kappa) \frac{g_1^2(E)}{g_2^3(E)} \tilde{G}_{ijkl} \tilde{\pi}^{ij} \tilde{\pi}^{kl} - \frac{\sqrt{\tilde{g}}}{2\kappa g_2(E)} \left(\tilde{R} + \frac{f(g_2^2(E) \tilde{R}) - 2\Lambda_c}{g_2^2(E)} \right) = 0, \quad (112)$$

where the “ \sim ” symbol means that we have rescaled every quantity like in section III. The Hamiltonian density (112) on the background (5) simplifies into

$$\mathcal{H}_m = f \left(2g_2^2(E) \frac{b'(r)}{r^2} \right) + 2 \frac{b'(r)}{r^2} - \frac{2\Lambda_c}{g_2^2(E)} = 0. \quad (113)$$

We can verify that the Schwarzschild solution leads to

$$g_2^2(E) f(0) = 2\Lambda_c. \quad (114)$$

Following the same steps of the previous section, we can write the graviton one loop contribution to the induced cosmological constant, whose form is

$$\frac{\Lambda^{f(R)}}{8\pi G} = -\frac{1}{3\pi^2} \sum_{i=1}^2 \int_{E^*}^{+\infty} E_i g_1(E) g_2(E) \frac{d}{dE_i} \sqrt{\left(\frac{E_i^2}{g_2^2(E)} - m_i^2(r) \right)^3} dE_i. \quad (115)$$

Note that with the assumption (102), the term $h(\mathcal{R}_{g_1 g_2})$ is absent and this simplifies technical calculations. Moreover with position (102), the scalar curvature R can be left unspecified. Let us see what happens when condition (55) holds. In this case Eq.(115) becomes

$$\frac{\Lambda^{f(R)}}{8\pi G} = -\frac{1}{3\pi^2} (I_+ + I_-), \quad (116)$$

where I_+ and I_- have been defined by Eqs.(61), (62) with the term $\sqrt{3 - \frac{1}{f'(0)}}$ dropped. We examine here the same proposals done in section III for $g_1(E/E_P)$ and $g_2(E/E_P)$. For the case a), essentially we can repeat what we have done in section III. Following the steps leading to Eq.(75), we find

$$\lim_{M \rightarrow 0} \frac{\Lambda}{8\pi G} = 0 = \frac{f(0)}{2\kappa} + \frac{3\pi - 8}{8\pi^2 c_1^2} E_P^4 \quad (117)$$

or

$$f(0) = -2 \frac{3\pi - 8}{\pi c_1^2} E_P^2. \quad (118)$$

We are therefore led to define

$$f(0) = \begin{cases} -(6\pi - 16)E_P^2 / (\pi c_1^2) & x = 0 \\ 0 & x > 0 \end{cases}. \quad (119)$$

This means that the correct Minkowski limit is reached when $M \rightarrow 0 \forall r \in [r_t, 5r_t/2]$. Concerning the case b), if one repeats the steps leading to Eq.(85), also in this case we conclude that the Minkowski limit cannot be reached for every r , because the value $r = 2MG$ with $M \rightarrow 0$ cannot vanish. As regards the case c), we find that Eq.(115) becomes:

$$\frac{\Lambda^{f(R)}}{8\pi G} = -\frac{I(z_\infty, x)}{8\pi^2} E_P^4, \quad (120)$$

where $I(z_\infty, x)$ has been defined in Eq.(93). Making the same steps of the case c) one finds

$$\frac{\Lambda^{f(R)}}{8\pi G} = \frac{\Lambda}{8\pi G} - \frac{1}{V16\pi G} \int_\Sigma d^3x \sqrt{g} f(g_2^2(E) \tilde{R}) = -\frac{E_P^4}{8\pi^2 c_3^4} I(1, y), \quad (121)$$

where $I(z_\infty, x) \rightarrow I(1, y)$. By imposing that the Minkowski limit be satisfied, one finds for $r > 2MG$

$$\frac{\Lambda}{8\pi G} - \frac{f(0)}{16\pi G} = -\frac{E_P^4}{8\pi^2 c_3^4} I(1, 0), \quad (122)$$

where for the Schwarzschild solution $\tilde{R} = 0$. Then one finds

$$f(0) = \frac{8E_P^2}{\pi c_3^4}, \quad (123)$$

where we have used the relationship $I(1, 0) = 4$. Plugging (123) into Eq.(121), we find

$$\frac{\Lambda}{8\pi G} = \frac{E_P^4}{8\pi^2 c_3^4} [4 - I(1, y)], \quad (124)$$

and the Minkowski limit is reached. Concerning the case $r = 2MG$, using Eqs.(98) and (99), we find

$$\frac{\Lambda}{8\pi G} - \frac{f(0)}{16\pi G} = -\frac{9E_P^8}{512\pi^2 M^4} \left[3\sqrt{2} - \ln(1 + \sqrt{2}) \right]. \quad (125)$$

By imposing that

$$\lim_{M \rightarrow 0} \frac{\Lambda}{8\pi G} = 0, \quad (126)$$

we find

$$f(0) = \frac{9E_P^6}{32\pi M^4} \left[3\sqrt{2} - \ln(1 + \sqrt{2}) \right] \quad (127)$$

and Eq.(121) becomes

$$\frac{\Lambda}{8\pi G} = \frac{E_P^4}{8\pi^2} \left[\frac{9E_P^4}{64M^4} \left[3\sqrt{2} - \ln(1 + \sqrt{2}) \right] - I\left(\frac{1}{c_3}, \frac{E_P}{2M} \sqrt{\frac{3}{2}}\right) \right]. \quad (128)$$

V. CONCLUSIONS

In this paper we have examined the effects of the combination of a $f(\mathcal{R})$ with Gravity's Rainbow. Since we have considered the contribution given by a distorted spherically symmetric background, we have used the identity $\mathcal{R} = R$ as can be seen from Eq.(7). This property allows to consider two different distortions coming from a $f(\mathcal{R})$ theory: the full covariant $4D$ scalar curvature \mathcal{R} and the spatial covariant $3D$ scalar curvature R . After having deduced how the respectively $f(\mathcal{R})$ models transform under Gravity's Rainbow, we have obtained the distorted WDW equations. In Refs.[9, 10] we have shown that the effect of Gravity's Rainbow is to offer an alternative method to keep under control UV divergences. However some of the proposals presented in Refs.[9, 10] have been discarded because the final induced cosmological constant had a bad Minkowski limit and a negative value. Here we have used the arbitrariness of $f(\mathcal{R})$ and $f(R)$ to correct the Minkowski limit and the negativity of Λ/κ . We draw to the reader's attention that for Minkowski limit we mean the following prescription

$$\lim_{M \rightarrow 0} \frac{\Lambda}{8\pi G} = 0 \quad (129)$$

and not

$$f(\mathcal{R})|_{\mathcal{R}=0} = 0 \quad \text{or} \quad f(R)|_{R=0} = 0. \quad (130)$$

It is important to remark that due to the complexity of transformations of $f(\mathcal{R})$ under Gravity's Rainbow, a general analysis is a very difficult task. This is not the case for a $f(R)$ theory, where in principle the form of the function can be kept quite general, even if with the choice of a metric of the kind (5), the complexity of the calculation can be further reduced. A considerable simplification of the model under examination is obtained for the Schwarzschild metric where $b(r) = 2MG$. With this choice, one finds $\mathcal{R} = R = 0$. Thus the general $f(\mathcal{R})$ theory reduces to an additional cosmological constant that can be used to shift the ZPE to the desired Minkowski limit. Indeed for both $f(\mathcal{R})$ and $f(R)$, we have found that what has been discarded in Refs.[9, 10], here can be accepted, provided one takes into account models admitting a discontinuity on the throat (horizon). We have to draw the reader's attention on the fact that even if $\mathcal{R} = R = 0$, this does not mean that the two proposals, namely $f(\mathcal{R})$ and $f(R)$ merge into one. Indeed the $f(\mathcal{R})$ theory generates into the one loop graviton operator (42) a term $h(\mathcal{R}_{g_1 g_2})$ containing $f'(\mathcal{R})$ which has its origin into the definition (19), while for a $f(R)$ model, the Lie derivative of the extrinsic curvature $\mathcal{L}_n K_{ij}$ is absent. In summary, for case a) we have

$$\left\{ \begin{array}{l} f(\mathcal{R})_{\mathcal{R}=0} = \begin{cases} f'(0) 2E_P^2 (3\pi - 8) / (\pi c_1^2) & x = 0 \\ 0 & x > 0 \end{cases} \\ f(R)_{|R=0} = \begin{cases} -(6\pi - 16)E_P^2 / (\pi c_1^2) & x = 0 \\ 0 & x > 0 \end{cases} \end{array} \right. , \quad (131)$$

while for case c), one gets

$$\left\{ \begin{array}{l} f(\mathcal{R})_{\mathcal{R}=0} = \begin{cases} 9f'(0) E_P^6 \sqrt{3 - \frac{1}{f'(0)}} (3\sqrt{2} - \ln(1 + \sqrt{2})) / (32\pi M^4) & r = 2MG \\ \sqrt{3 - \frac{1}{f'(0)}} f'(0) E_P^2 / (\pi c_3^4) & r > 2MG \end{cases} \\ f(R)_{|R=0} = \begin{cases} 9E_P^6 [3\sqrt{2} - \ln(1 + \sqrt{2})] / (32\pi M^4) & r = 2MG \\ 8E_P^2 / (\pi c_3^4) & r > 2MG \end{cases} \end{array} \right. . \quad (132)$$

We recall that the case b) did not produce the correct Minkowski limit for both $f(\mathcal{R})$ and $f(R)$ theories and therefore it has been discarded. At this point one question must be posed: what is the impact of imposing the Minkowski limit on the behavior of the “*cosmological constant*”. First of all, one has to note that in this approach one finds that what is found is a “*dynamical cosmological constant*” which is variable with the radial coordinate r instead of a time coordinate t . Although this is not a result due to the combined effect of Gravity's Rainbow and $f(\mathcal{R})$ or $f(R)$ models, because a “*dynamical cosmological constant*” was introduced in Ref.[14] and subsequently in Ref.[9], we have to remark that such a combination enlarges the family of models that potentially can explain the behavior of the

cosmological constant in the different epochs. For example, the model (65) discussed also in Ref.[9] without the help of a $f(\mathcal{R})$ or $f(R)$ modification needed two different rainbow's functions matching at some space point to have a correct Minkowski limit for every Schwarzschild mass M . Therefore the combination of Gravity's Rainbow and $f(\mathcal{R})$ or $f(R)$ seems to have the right properties to extract the necessary information about the cosmological constant. We have to stress that all we need is a finite not vanishing value of $f(\mathcal{R})$ or $f(R)$ when $\mathcal{R} = R = 0$ and a finite value of $f'(\mathcal{R})$ for $\mathcal{R} = 0$. This does not mean that every proposal can be accepted. For instance models of the form

$$f(\mathcal{R}) = \mathcal{R} \pm \frac{\mu^{2(n+1)}}{\mathcal{R}^n} \quad \text{or} \quad f(R) = R \pm \frac{\mu^{2(n+1)}}{R^n} \quad n \geq 1 \quad (133)$$

cannot be taken as a viable examples because of their singularity in the Schwarzschild background. We arrive to the same conclusion to proposals of the form[24]

$$f(\mathcal{R}) = A \ln(\alpha \mathcal{R}) \quad \text{or} \quad f(R) = B \ln(\alpha R), \quad (134)$$

where A and B are appropriate constants needed to reestablish the correct dimensions. At the present stage, we do not know if this is a failure of our proposal or an indication helping to select the various models. We have also to remark that the considerations done in (133) and in (134) are independent on the Gravity's Rainbow scheme, at least for the Schwarzschild background. On the other hand proposals like

$$f(\mathcal{R}) = A \exp(-\alpha \mathcal{R}) \quad \text{or} \quad f(R) = B \exp(-\alpha R) \quad (135)$$

have the correct properties to shift the ZPE solution to the desired Minkowski value. It is interesting to observe that usually one constrains $f(\mathcal{R})$ or $f(R)$ to have the flatness property (130) and

$$\lim_{\mathcal{R} \rightarrow \infty} f(\mathcal{R}) = -\Lambda_{Planck} \quad \text{or} \quad \lim_{R \rightarrow \infty} f(R) = -\Lambda_{Planck} \quad (136)$$

to have inflation, where Λ_{Planck} is of Planckian size. In our approach the situation seems to be reversed because the condition (129) generates a big $f(0)$ in both $4D$ and $3D$ to compensate the effects of ZPE. At first glance one could conclude that computing ZPE with the help of a $f(\mathcal{R})$ combined to Gravity's Rainbow is not so different to computing ZPE with the help of a $f(R)$ theory combined with Gravity's Rainbow since the main difference is in $f'(\mathcal{R})$. However all the considerations done hitherto are about the Schwarzschild metric, from one side and from the other side when one introduces also boundary term this difference is more marked. Indeed the boundary action for a $f(\mathcal{R})$ model is[25]

$$\int_{\partial \mathcal{M}} d^3x \sqrt{|h|} f'(\mathcal{R}) K, \quad (137)$$

where K is the trace of the second fundamental form, $\partial \mathcal{M}$ is the boundary of the manifold \mathcal{M} and h is the induced metric on $\partial \mathcal{M}$. Of course when $f(\mathcal{R}) = \mathcal{R}$, the boundary term reduces to the usual case of General Relativity which also coincides with the $f(R)$ proposal, namely

$$\int_{\partial \mathcal{M}} d^3x \sqrt{|h|} K. \quad (138)$$

This simple but relevant difference opens up a window on the way in which some problems like black hole pair creation, entropy computation can be computed in the respective schemes.

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